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Extreme value theory and high quantile convergence

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In this paper we raise some issues concerning estimation of high quantiles and shortfalls using extreme value theory (EVT). We demonstrate that for a wide class of distribution, EVT does not lead to uniform relative quantile convergence. Further we show that, in general, EVT does not lead to mean convergence.

1 Introduction

The advanced measurement approach (AMA) to the Pillar I modeling of operational risk requires estimation of high quantiles for loss distributions. A number of articles have analyzed the application of extreme value theory (EVT) to the estimation of high quantiles for heavy tailed distributions (cf Embrechts 2000; McNeil 1997; McNeil *et al* 2006). Although many authors have observed difficulties in application of the method to real or even simulated data, those difficulties were attributed to data pollution, uncertainty in selection of the threshold, or an insufficient amount of data (cf Embrechts 2000; Nešlehová *et al* 2006; Mignola and Ugoccioni 2005). In this article we show that for a wide class of distributions (for example, log gamma) EVT cannot estimate high quantiles or shortfalls as it does not provide a sufficiently good approximation to the estimated distribution. The reason why EVT cannot be used to estimate high quantiles for a wide class of distributions is simple: EVT guarantees convergence in distribution only. Convergence in distribution, however, does not imply uniform convergence for quantiles, convergence for mean, or convergence for shortfalls. Hence, when applying EVT method to a distribution with finite mean, it can result in an approximation which has infinite mean, significantly different high quantiles and shortfalls.

The article is organized as follows: in Section 2 we recall the EVT tail approximation method; in Section 3 we discuss convergence required for quantiles and shortfall estimation; in Section 4 we give an example for which EVT approximation does lead to a good estimation of high quantiles; in Section 5 we give an example of a family of distributions for which EVT cannot estimate high quantiles and shortfalls; in Section 6 we give an example where EVT leads to a distribution with infinite mean while the estimated distribution has finite mean; finally, in Section 7, some remarks and suggestions conclude the article.

2 EVT tail approximation

In this section we review the main definitions and results of EVT tail approximation method (cf McNeil *et al* 2006).

For a cumulative distribution function (cdf) F, let us introduce the following notations:

(a) $\bar{F}(x) = 1 - F(x)$

(b) $F_u(x) = \Pr[X - u \leqslant x \mid X > u] = \dfrac{F(x+u) - F(u)}{1 - F(u)}$

DEFINITION 2.1 A positive function L is slowly varying if, for any $t > 0$,

$$\lim_{x\to\infty}\left[\frac{L(tx)}{L(x)}\right] = 1$$

EVT can be applied to regularly varying (RV) cdfs, which are introduced in the following.

DEFINITION 2.2 For $\xi > 0$, $F \in RV_\xi$ if for some slowly varying function L

$$\bar{F}(x) = 1 - F(x) = x^{-1/\xi}L(x)$$

For $\xi > 0$, the cdf of a generalized Pareto distribution (GPD) is given by

$$G_{\xi,\beta}(x) = 1 - \left(1 + \frac{\xi x}{\beta}\right)^{-1/\xi}$$

THEOREM 2.3 (Pickands–Balkema–de Haan) *$F \in RV_\xi$ if and only if there exists a function $\beta(u)$ such that*

$$\lim_{u\to\infty} \sup_{0\leqslant x<\infty} |F_u(x) - G_{\xi,\beta(u)}(x)| = 0$$

The EVT uses Theorem 2.3 in order to find an approximation to the tail of the loss data. Given a loss data set, one selects a threshold u for which a GPD is a good approximation of the tail. The losses below the threshold are modeled using an empirical distribution; the losses above the threshold are modeled using the GPD.

In this paper we will assume that not only the losses but also their underlying distribution F itself are known and will check how the EVT model performs under such idealized conditions.

DEFINITION 2.4 For a cdf F, denote by $\mathrm{EVT}[F, u]$ the approximation to F obtained by using the EVT tail:

$$\mathrm{EVT}[F, u](x) = \begin{cases} F(x) & x \leqslant u \\ F(u) + (1 - F(u))G_{\xi,\beta(u)}(x - u) & x \geqslant u \end{cases}$$

3 Quantile convergence

Theorem 2.3 implies convergence in distribution of F_u to $G_{\xi,\beta(u)}$ when the threshold u goes to infinity. The convergence in distribution is a weak form of convergence. In particular, it does not guarantee convergence in mean or convergence for quantiles.

For risk measurement applications involving estimation of quantiles and/or shortfalls, one requires that the relative error in the estimation of the *quantiles* is small. Hence, for such applications, the following convergence is required from EVT approximation.

DEFINITION 3.1 EVT approximation leads to uniform relative quantile (URQ) convergence if, for any $\varepsilon > 0$, there exists a threshold u_0 such that for any $u \geqslant u_0$ and any quantile q,

$$\begin{aligned}(1-\varepsilon)\ \text{Quantile}(\text{EVT}[F, u], q) &\leqslant \text{Quantile}(F, q)\\ &\leqslant (1+\varepsilon)\ \text{Quantile}(\text{EVT}[F, u], q)\end{aligned}$$

Denote by $EF_{\pm\varepsilon;u}$ the cdfs corresponding to

$$(1 \pm \varepsilon)\ \text{Quantile}(\text{EVT}[F, u], q)$$

LEMMA 3.2 *In case of URQ convergence, the following hold:*

(1) $EF_{+\varepsilon;u}(x) \leqslant F(x) \leqslant EF_{-\varepsilon;u}(x)$

(2) $EF_{\pm\varepsilon;u}(x) = EVT[F, u]\left(\dfrac{x}{1 \pm \varepsilon}\right)$

PROOF Both statements follow from the properties of the inverse function. □

LEMMA 3.3 *URQ convergence implies convergence in mean.*

PROOF The mean of a cdf G can be expressed as follows:

$$\text{Mean(G)} = \int_0^1 \text{Quantile}(G, q)\, \mathrm{d}q$$

Therefore, URQ convergence of EVT approximation implies

$$(1-\varepsilon)\ \text{Mean}(\text{EVT}[F, u]) \leqslant \text{Mean}(F) \leqslant (1+\varepsilon)\ \text{Mean}(\text{EVT}[F, u])$$ □

The following proposition gives a necessary condition for URQ convergence of EVT approximation.

PROPOSITION 3.4 *For $F \in RV_\xi$, EVT approximation leads to the URQ convergence only if*

$$0 < \lim_{x\to\infty} \bar{F}(x)x^{1/\xi} < \infty$$

PROOF By Lemma 3.2(1) we have

$$\lim_{x\to\infty} \overline{EF}_{-\varepsilon;u}(x)x^{1/\xi} \leqslant \lim_{x\to\infty} \bar{F}(x)x^{1/\xi} \leqslant \lim_{x\to\infty} \overline{EF}_{+\varepsilon;u}(x)x^{1/\xi}$$

Applying Lemma 3.2(2) and Definition 2.4, we obtain

$$\begin{aligned}\lim_{x\to\infty} \overline{EF}_{\mp\varepsilon;u}(x)x^{1/\xi} &= \lim_{x\to\infty} \overline{\mathrm{EVT}}[F, u]\left(\frac{x}{1\mp\varepsilon}\right)x^{1/\xi} \\ &= (1 - F(u)) \lim_{x\to\infty} G_{\xi,\beta(u)}\left(\frac{x}{1\mp\varepsilon} - u\right)x^{1/\xi} \\ &= (1 - F(u))\left(\frac{\beta(u)(1\mp\varepsilon)}{\xi}\right)^{1/\xi}\end{aligned}$$

Therefore,

$$\begin{aligned}0 < (1 - F(u))\left(\frac{\beta(u)(1-\varepsilon)}{\xi}\right)^{1/\xi} &\leqslant \lim_{x\to\infty} \bar{F}(x)x^{1/\xi} \\ &\leqslant (1 - F(u))\left(\frac{\beta(u)(1+\varepsilon)}{\xi}\right)^{1/\xi} < \infty \qquad \square\end{aligned}$$

4 Example of EVT approximation with URQ convergence

This example illustrates that for some distributions, EVT approximation does lead to URQ convergence. Following Nešlehová *et al*, consider a cdf given by a mixture of two Pareto distributions (cf Nešlehová *et al* (2006, Example 2.2)):

$$F(x) = 1 - 0.9x^{-1.4} - 0.1x^{-0.6}$$

To apply EVT we compute

$$\bar{F}_u(x) = \frac{9u^{1.4}}{9 + u^{0.8}}(u + x)^{-1.4} + \frac{u^{1.4}}{9 + u^{0.8}}(u + x)^{-0.6}$$

$\bar{F}_u(x)$ can be approximated by a GPD

$$\bar{G}_u(x) = \frac{u^{1.4}}{9 + u^{0.8}}(u + x)^{-0.6}$$

LEMMA 4.1 *For $u \to \infty$, $F_u(x)$ converges to $G_u(x)$ in the distribution sense.*

PROOF We have

$$\begin{aligned}|F_u(x) - G_u(x)| &= |\bar{F}_u(x) - \bar{G}_u(x)| \\ &= \frac{9u^{1.4}}{9 + u^{0.8}}(u + x)^{-1.4} \leqslant \frac{9u^{1.4}}{9 + u^{0.8}}u^{-1.4} = \frac{9}{9 + u^{0.8}} \qquad \square\end{aligned}$$

LEMMA 4.2 *For $u \to \infty$, $F_u(x)$ converges to $G_u(x)$ in the URQ sense.*

PROOF One easily shows that $\bar{F}_u(x)$ can be bounded by multiples of $\bar{G}_u(x)$ as follows

$$\bar{G}_u(x) \leqslant \bar{F}_u(x) \leqslant \left(1 + \frac{9}{u^{0.8}}\right)\bar{G}_u(x)$$

Therefore,

$$\text{Quantile}(\bar{G}_u,\, p) \geqslant \text{Quantile}(\bar{F}_u,\, p) \geqslant \text{Quantile}\left(\left(1 + \frac{9}{u^{0.8}}\right)\bar{G}_u(x),\, p\right)$$

Taking into account the equations

$$\text{Quantile}(\bar{G}_u,\, p) = \left(\frac{p(9 + u^{0.8})}{u^{1.4}}\right)^{-5/3}$$

$$\begin{aligned}\text{Quantile}\left(\left(1 + 9/u^{0.8}\right)\bar{G}_u(x),\, p\right) &= \left(\frac{p(9 + u^{0.8})}{(1 + \frac{9}{u^{0.8}})u^{1.4}}\right)^{-5/3} \\ &= \left(1 + \frac{9}{u^{0.8}}\right)^{5/3}\left(\frac{p(9 + u^{0.8})}{u^{1.4}}\right)^{-5/3} \\ &= \left(1 + \frac{9}{u^{0.8}}\right)^{5/3}\text{Quantile}(\bar{G}_u,\, p)\end{aligned}$$

we conclude that

$$\left(1 + \frac{9}{u^{0.8}}\right)^{5/3}\text{Quantile}(\bar{G}_u(x),\, p) \leqslant \text{Quantile}(\bar{F}_u,\, p) \leqslant \text{Quantile}(\bar{G}_u,\, p)$$

which proves URQ convergence when $u \to \infty$. □

5 Example of EVT approximation without URQ convergence

This example illustrates that for many RV functions, EVT approximation does not lead to URQ convergence.

Consider random variables with cdfs given by

$$F_{a,b}(x) = 1 - \frac{\log^a(x)}{x^b}$$

where $b > 0,\ a \neq 0$.

It can be shown that for a > 0 the above family of distributions is asymptotically close to log gamma distributions and therefore all the result below applies to log gamma family as well.

LEMMA 5.1 *We have the following:*

(1) $F_{a,b} \in RV_{1/b}$;
(2) *EVT approximation does not lead to URQ convergence for* $F_{a,b}$.

PROOF To prove (1), we need to show that $\log^a(x)$ is a slowly varying function. Indeed we have

$$\lim_{x\to\infty} \frac{\log^a(t\,x)}{\log^a(x)} = \lim_{x\to\infty} \left(\frac{\log(t) + \log(x)}{\log(x)}\right)^a = 1$$

To prove (2), note that

$$\lim_{x\to\infty} \bar{F}_{a,b}(x)x^b = \begin{cases} \lim\limits_{x\to\infty} \log^a(x) = \infty & \text{for a } > 0 \\ \lim\limits_{x\to\infty} \log^a(x) = 0 & \text{for a } < 0 \end{cases}$$

Hence, the necessary condition for URQ convergence stated in Proposition 3.4 is violated and, therefore, EVT approximation cannot lead to URQ convergence. □

6 Example of EVT approximation without mean convergence

In this example we construct a distribution with finite mean for which the EVT method produces an approximation with infinite mean.

To construct the example, consider a random variable X with the following cdf:

$$F_{1,-2}(x) = 1 - \frac{1}{x\log^2(x)}, \quad x \geqslant e$$

The distribution belongs to the distribution family considered in Example 2 and, thus, the EVT method does not lead to URQ convergence.

LEMMA 6.1 *$F_{1,-2}$ is a distribution with finite mean.*

PROOF The mean of F can be computed as follows:

$$\begin{aligned} Mean(F) &= \int_0^\infty (1 - F_{1,-2}(x))\,\mathrm{d}x \\ &= e + \int_e^\infty (1 - F_{1,-2}(x))\,\mathrm{d}x = e + \frac{1}{\log(e)} = e + 1 \end{aligned}$$

□

COROLLARY 6.2 *The following statements hold.*

(1) *EVT approximates the distribution $F_{1,-2}$ with finite mean by a distribution with infinite mean.*
(2) *EVT approximation does not estimate mean, shortfalls, or high quantiles of $F_{1,-2}$ correctly.*

PROOF To prove (1), note that $F_{1,-2} \in RV_1$ and thus EVT approximates the tail of $F_{1,-2}$ with a GPD with $\xi = 1$ which has infinite mean.

Statement (2) follows from Lemmas 3.3 and 6.1. □

7 Conclusions

The examples in Sections 4–6 illustrate that for some distributions EVT approximation leads to uniform relative quantile convergence whilst for others it does not. Similarly, convergence may or may not exist for shortfalls and expected values. It would be interesting to develop statistical tests to test when for a given data set the EVT method leads to uniform quantile convergence. One can also probably prove that EVT approximation leads to convergence for some quantiles and use its convergence property only for those quantiles. Meanwhile, for the purpose of estimation of quantiles and shortfalls, it seems to be the best approach to fit not only the GPD but also other types of distributions (for example, log gamma).

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